AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

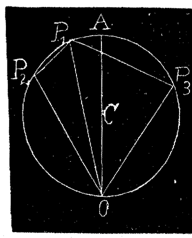
SOLUTIONS OF PROBLEMS.

30. Proposed by F. P. MATZ, M. A., M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all the triangles which can be inscribed in a given circle.

I. Solution by the PROPOSER.

Let P_1OP_2 be any inscribed triangle; and through O draw any diameter OA. Two cases have now to be considered: (1), the triangle may lie *wholly* on one side of the diameter OA; (2), the triangle may lie *partly* on one side of the diameter OA.



I. Put $OA=2r$, $\angle AOP_1=\phi$, and $\angle AOP_2=\theta$; then $OP_1=2r\cos\phi$, $OP_2=2r\cos\theta$, and the area of the $\triangle$, P_1OP_2, $=A'$, $=2r^2\cos\phi\cos\theta\sin(\phi-\theta)$. Hence the average area of the triangles in this case, is

$$A_1=\int_0^{\frac{1}{2}\pi}\int_0^{\phi}A'd\phi d\theta\div\int_0^{\frac{1}{2}\pi}\int_0^{\phi}d\phi d\theta=\frac{8r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}\phi\sin\phi\cos\phi d\phi$$

$$=\frac{r^2}{\pi^2}\Big[\sin2\phi-2\phi\cos2\phi\Big]_0^{\frac{1}{2}\pi}=\frac{r^2}{\pi}\ldots\ldots\ldots\ldots(1).$$

II. Put $\angle AOP_3=\psi$; then the area of the triangle P_2OP_3, $=A''$, $=2r^2\cos\theta\cos\psi\sin(\theta+\psi)$. Hence the average area of the triangles in this case, is

$$A_2=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}A''d\theta d\psi\div\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}d\theta d\psi=\frac{8r^2}{\pi^2}\Big[\frac{\pi}{4}\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta d\theta$$

$$+\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta\Big]=\frac{8r^2}{\pi^2}\Big[\frac{\pi}{4}\times\tfrac{1}{2}+\tfrac{1}{2}\times\frac{\pi}{4}\Big]=\frac{2r^2}{\pi}\ldots\ldots\ldots\ldots(2).$$

Hence the *required* average area becomes

$$A=\tfrac{1}{2}(A_1+A_2)=3r^2/2\pi\ldots\ldots\ \ldots\ldots\ldots\ \ldots..(3).$$

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

We readily get the area of triangle

$$=\frac{R^2}{2}\{\sin2A+\sin2B+\sin2C\},$$

which, by virtue of the relation $A+B+C=\pi$, reduces to

$$\frac{R^2}{2}\{\sin 2A+\sin 2B-\sin 2(A+B)\}.$$

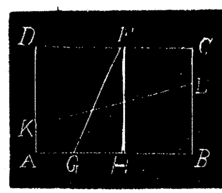
$$\therefore\ \text{Average area}=\frac{R^2}{2}\,\frac{\int_0^{\pi}\int_0^{\pi-A}\{\sin 2A+\sin 2B-\sin 2(A+B)\}dAdB}{\int_0^{\pi}\int_0^{\pi-A}dAdB}=\frac{3R^2}{2\pi}.$$

31. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the average length of a line drawn across the opposite sides of a rectangle, length l and breadth b.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and the PROPOSER.

Let $ABCD$ be the rectangle, FG the random line. Let $AB=l$, $BC=b$, $AH=x$, $AG=y$.



Then $FG=\{b^2+(x-y)^2\}^{\frac{1}{2}}$.

The limits of x are 0 and l; of y, 0 and x.

Hence the required average area is

$$\Delta=\frac{\int_0^l\int_0^x\{b^2+(x-y)^2\}^{\frac{1}{2}}\,dxdy}{\int_0^l\int_0^x dxdy}$$

$$=\frac{2}{l^2}\int_0^l\int_0^x\{b^2+(x-y)^2\}^{\frac{1}{2}}\,dxdy$$

$$=\frac{1}{l^2}\int_0^l\{x(b^2+x^2)^{\frac{1}{2}}+b^2\log[x+(b^2+x^2)^{\frac{1}{2}}]-b^2\log b\}dx$$

$$=\frac{1}{3l^2}(l^2+b^2)^{\frac{3}{2}}+\frac{b^2}{l}\log\{l+(l^2+b^2)^{\frac{1}{2}}\}-\frac{b^2}{l}\log b-\frac{1}{l^2}(l^2+b^2)^{\frac{1}{2}}-\frac{b^3}{3l^2}+\frac{b}{l^2}.$$

For the line KL, we get, by writing l for b and b for l,

$$\Delta_1=\frac{1}{3b^2}(l^2+b^2)^{\frac{3}{2}}+\frac{l^2}{b}\log\{b+(l^2+b^2)^{\frac{1}{2}}\}-\frac{l^2}{b}\log l-\frac{1}{b^2}(l^2+b^2)^{\frac{1}{2}}-\frac{l^3}{3b^2}+\frac{l}{b^2}.$$

Cor. I. If $l=b$, $\Delta=\frac{1}{3}(2l\sqrt{2})+l\log(1+\sqrt{2})-\frac{1}{l}\sqrt{2}-\frac{1}{3}l+\frac{1}{l}$.

Cor. II. If $l=b=1$, $\Delta=\frac{1}{3}(2-\sqrt{2})+\log(1+\sqrt{2})$, which is the same result as given in *Williamson's Integral Calculus*, page 409.

Also solved by *F. P. MATZ*.